Solve $a \sin x + b \cos x = c$

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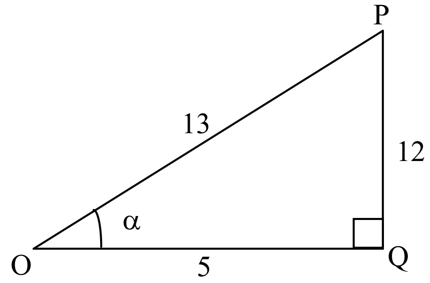
Question

Solve : $12 \sin x + 5 \cos x = 4$

Method 1



Construct a triangle OPQ with $\tan \alpha = \frac{12}{5}$.

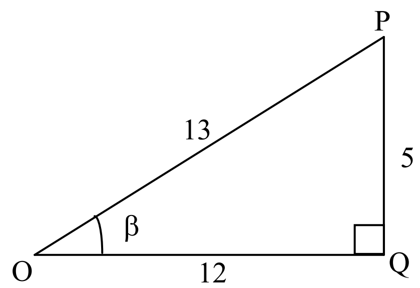
$\therefore \quad \alpha \approx 67.380^\circ$.

Also, $OP = \sqrt{5^2 + 12^2} = 13$

Now,
$$12 \sin x + 5 \cos x = 4$$
$$(13 \sin \alpha) \sin x + (13 \cos \alpha) \cos x = 4$$
$$13 (\cos x \cos \alpha + \sin x \sin \alpha) = 4$$
$$13 \cos (x - \alpha) = 4$$
$$\cos(x - \alpha) = \frac{4}{13}$$
$$x - 67.380^\circ \approx 360^\circ n \pm 72.080^\circ$$
$$x \approx 360^\circ n \pm 72.080^\circ + 67.380^\circ$$

$\therefore \quad x \approx 360^\circ n + 139.460^\circ$ or $x \approx 360^\circ n - 4.700^\circ$,where n is an integer . …. (1)

Method 2



Construct a triangle OPQ with $\tan \beta = \frac{5}{12}$.

$\therefore \quad \alpha \approx 22.620^\circ$.

Also, $OP = \sqrt{5^2 + 12^2} = 13$

Now,
$$12 \sin x + 5 \cos x = 4$$
$$(13 \cos \beta) \sin x + (13 \sin \beta) \cos x = 4$$
$$13 (\sin x \cos \beta + \cos x \sin \beta) = 4$$
$$13 \sin (x + \beta) = 4$$
$$\sin(x + \beta) = \frac{4}{13}$$
$$x + 22.620^\circ \approx 180^\circ n + (-1)^n 17.920^\circ$$

$x \approx 180^\circ n + (-1)^n 17.920^\circ - 22.620^\circ$, where n is an integer . …. (2)

To show that the solution set (2) is the same as the solution set (1), we need to divide the solution into **odd** and **even** cases :

(a) $n = 2m$, $x \approx 180^\circ(2m) + 17.920^\circ - 22.620^\circ = 360^\circ m - 4.700^\circ$

(b) $n = 2m + 1$, $x \approx 180^\circ(2m+1) - 17.920^\circ - 22.620^\circ = 360^\circ m + 139.460^\circ$, where m is an integer.

Method 3 (t – method)

Let $t = \tan\frac{x}{2}$.

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\cos^2\frac{x}{2} = \frac{2\tan\frac{x}{2}}{\sec^2\frac{x}{2}} = \frac{2t}{1+t^2}$$

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \left(1 - \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}\right)\cos^2\frac{x}{2} = \frac{1-\tan^2\frac{x}{2}}{\sec^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}$$

Now, the equation : $12\sin x + 5\cos x = 4$

$$12\left(\frac{2t}{1+t^2}\right) + 5\left(\frac{1-t^2}{1+t^2}\right) = 4$$

$$12(2t) + 5(1-t^2) = 4(1+t^2)$$

$$9t^2 - 24t - 1 = 0$$

$$t = \frac{4 \pm \sqrt{17}}{3}$$

$\therefore \quad \tan\frac{x}{2} \approx 2.7077$ or $\tan\frac{x}{2} \approx -0.041035$

$\therefore \quad \frac{x}{2} \approx 180^\circ n + 69.730^\circ$ or $\frac{x}{2} \approx 180^\circ n - 2.3500^\circ$

$\therefore \quad x \approx 360^\circ n + 139.460^\circ$ or $x \approx 360^\circ n - 4.700^\circ$, where n is an integer . …. (3)

Method 4 (unsatisfactory)

$$12\sin x + 5\cos x = 4$$

$$12\sin x \pm 5\sqrt{1-\sin^2 x} = 4$$

$$\pm 5\sqrt{1-\sin^2 x} = 4 - 12\sin x$$

$$25(1-\sin^2 x) = 16 - 96\sin x + 144\sin^2 x$$

$$169\sin^2 x - 96\sin x - 9 = 0$$

$$\sin x = \frac{48 \pm 15\sqrt{17}}{169} \approx 0.64998 \quad \text{or} \quad -0.081932$$

$x \approx 180^\circ n + (-1)^n 40.540^\circ$ or $x \approx 180^\circ n + (-1)^n(-4.700^\circ)$ …. (4)

Method 4 is unsatisfactory since solution set (4) gives a larger solution set than (1), (2) or (3). Readers please explain. What solutions should be rejected ? How to reject ?

Explanation

Squaring equation creates roots! That is why Method 4 is not recommended.

Finding the redundant roots is tedious:

In (4),

(a) When $x \approx 180^\circ n + (-1)^n 40.540^\circ$.

(i) n = 2m, $x \approx 360^\circ m + 40.540^\circ$

(ii) n = 2m + 1, $x \approx 360^\circ m + 139.460^\circ$

(b) When $x \approx 180^\circ n + (-1)^n (-4.700^\circ)$

(i) n = 2m, $x \approx 360^\circ m - 4.700^\circ$

(ii) n = 2m + 1, $x \approx 360^\circ m + 184.700^\circ$

Therefore (4) is equivalent to :

$$\begin{cases} x \approx 360^\circ m + 40.540^\circ & \text{....(4.1)} \\ x \approx 360^\circ m + 139.460^\circ & \text{....(4.2)} \\ x \approx 360^\circ m - 4.700^\circ & \text{....(4.3)} \\ x \approx 360^\circ m + 184.700^\circ & \text{....(4.4)} \end{cases}$$

(4.2) and (4.3) are good solutions.

(4.1) and (4.4) should be rejected.

For (4.1), 12 sin x + 5 cos x

$$\begin{aligned} &= 12\sin(360^\circ m + 40.540^\circ) + 5\cos(360^\circ m + 40.540^\circ) \\ &= 12\sin 40.540^\circ + 5\cos 40.540^\circ \\ &\approx 12(0.64998) + 5(0.75995) \\ &\approx 11.59951 \\ &\neq 4 \end{aligned}$$

$\therefore$ $x \approx 360^\circ m + 40.540^\circ$ does not satisfy the equation 12 sin x + 5 cos x = 4, and should be rejected.

Similarly, for (4.4), $x \approx 360^\circ m + 184.700^\circ$ should be rejected.